MATH 2230A

Tutorial 2 21-9-2016

Topics: ① CR-equation and related example
② CR-equation in polar form
③ Harmonic function

CR-equation and related example

Thm: A function $f: \Omega \subseteq \mathbb{C} \Rightarrow \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$ iff. f is real differentiable and satisfies the CR-equation:

- if $f = u + iv$, then $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

Recall: For $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (u(x,y), v(x,y))$, the Jacobian matrix of f is defined by

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

In particular, for complex differentiable $f: \Omega \to \mathbb{C}$,

① The Jacobian matrix has the form

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \overset{\text{CR-eqtn}}{=\!=} \begin{pmatrix} p & -q \\ q & p \end{pmatrix},$$

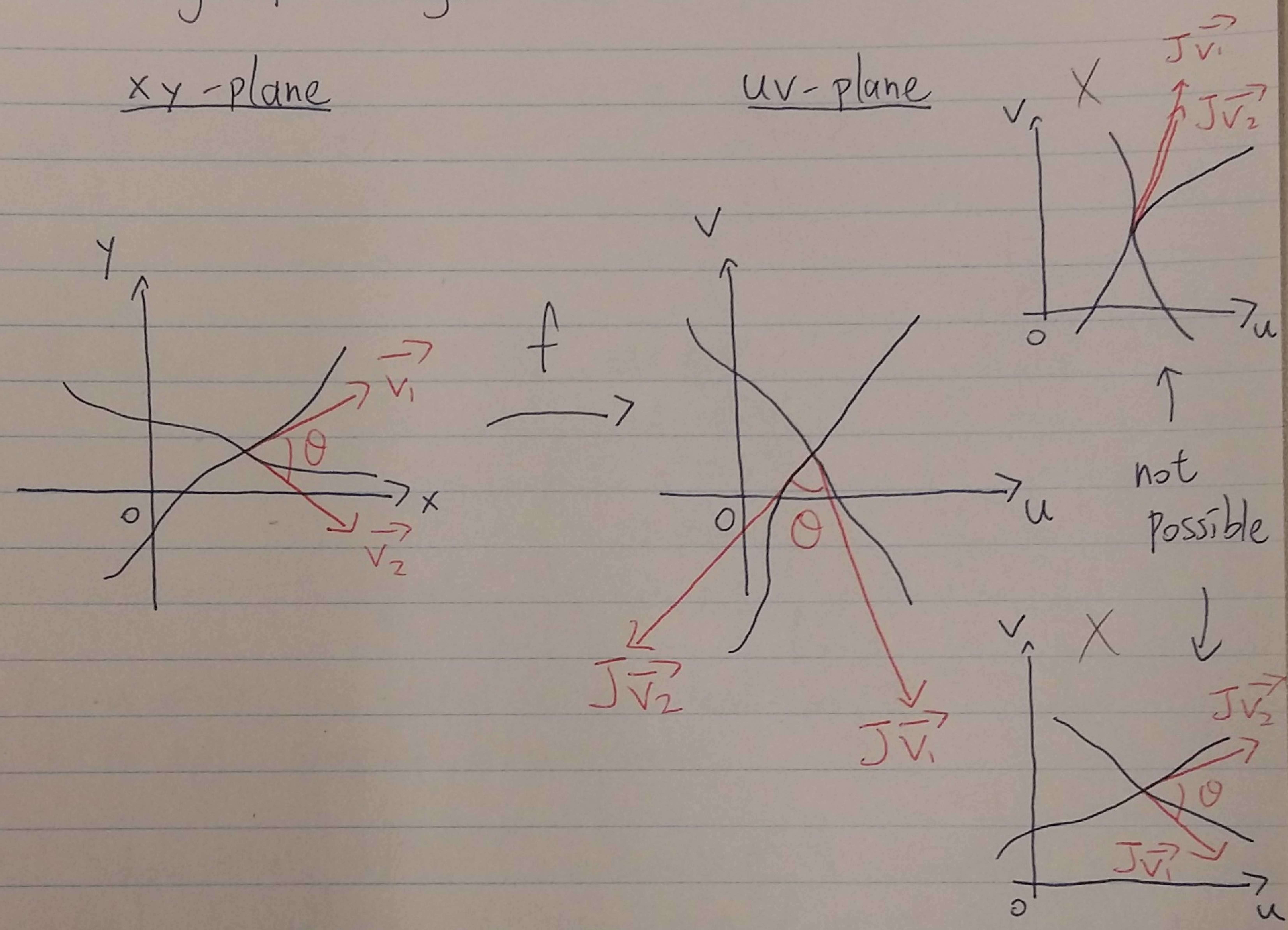
with $p = u_x = v_y$ and $q = -u_y = v_x$.

Moreover,

$$J = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} = \sqrt{p^2+q^2} \begin{pmatrix} \frac{p}{\sqrt{p^2+q^2}} & -\frac{q}{\sqrt{p^2+q^2}} \\ \frac{q}{\sqrt{p^2+q^2}} & \frac{p}{\sqrt{p^2+q^2}} \end{pmatrix}$$

$$\Rightarrow J = \underbrace{\sqrt{p^2+q^2}}_{\text{Enlargement}} \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\text{Rotation}}, \quad \begin{cases} \cos\alpha = \dfrac{p}{\sqrt{p^2+q^2}} \\ \sin\alpha = \dfrac{q}{\sqrt{p^2+q^2}} \end{cases}$$

This means complex differentiability implies conformality.
(i.e. angle preserving)



② Since $f'(z) = u_x + iv_x$, $|f'(z)| = \sqrt{u_x^2 + v_x^2} = \det(J)$.

$\Rightarrow$ $|f'(z)|$ = magnitude of enlargement

Example: 1) Show that the function

$$f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$$
$$z \mapsto \frac{1}{\bar{z}}$$

is not complex differentiable on its domain.

Ans: $f(z) = \dfrac{1}{\bar{z}} = \dfrac{z}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

So we have

$$u(x,y)=\frac{x}{x^2+y^2} \text{ and } v(x,y)=\frac{y}{x^2+y^2}.$$

$$\Rightarrow \begin{cases} u_x=\dfrac{y^2-x^2}{(x^2+y^2)^2}, & v_x=\dfrac{-2xy}{(x^2+y^2)^2} \\ u_y=\dfrac{-2xy}{(x^2+y^2)^2} & v_y=\dfrac{x^2-y^2}{(x^2+y^2)^2} \end{cases}$$

CR-eqtns holds $\Leftrightarrow \begin{cases} u_x=v_y \\ u_y=-v_x \end{cases}$

$\Leftrightarrow \begin{cases} x^2=y^2 \\ xy=0 \end{cases}$

$\Leftrightarrow x=0=y$

As a result, CR-eqtns do not hold in $\mathbb{C}\setminus\{0\}$ and f is not complex differentiable.

2) Show that if $f:\Omega\subseteq\mathbb{C}\to\mathbb{C}$ is complex differentiable and $|f(z)|=c$ $\forall z\in\Omega$, then f must be a constant function.

Ans: Write $f=u+iv$. We have $u^2+v^2=$ constant.
Diff. both sides w.r.t. x and y, we have

$$\begin{cases} 2uu_x+2vv_x=0 \\ 2uu_y+2vv_y=0 \end{cases} \Rightarrow \begin{cases} uu_x=-vv_x \\ uu_y=-vv_y. \end{cases}$$

By CR-eqtn, we have

$$\begin{cases} uu_x=vu_y & \text{— (1)} \\ uu_y=-vu_x & \text{— (2)} \end{cases}$$

From (1), we have $u(vu_x)=v^2u_y$.

$$\xRightarrow{\text{By (2)}} -u^2u_y=v^2u_y$$

$$\Rightarrow (u^2+v^2)u_y=0$$

If $u^2+v^2=0$, then $u=0=v$ and so $f\equiv 0$ is a constant function. Otherwise we have $u_y=0$.

Similarly, from (1), $u^2 u_x = uvu_y$

$$\begin{aligned}
&\Rightarrow \quad u^2 u_x = -v^2 u_x \\
&\Rightarrow \quad (u^2+v^2)u_x = 0 \\
&\Rightarrow \quad u_x = 0
\end{aligned}$$

By CR-eqtn, we also have $v_x = v_y = 0$.
Hence f is a constant function.

<u>Remark</u>: It is not true in real case.
For example, $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$ defined on $\mathbb{R}^2\setminus\{0\}$
is non-constant function with $|f|=1$.

<u>CR-eqtn in polar form</u>

$$(x,y) \longleftrightarrow (r,\theta) \qquad (z\neq 0)$$

$$\begin{aligned}
f: \Omega &\to \mathbb{C} \\
(x,y) &\mapsto (u,v)
\end{aligned}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \xRightarrow[\text{Rule}]{\text{Chain}} \begin{array}{ll}
\text{(i)} & u_r = u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta \\
\text{(ii)} & u_\theta = u_x x_\theta + u_y y_\theta = -ru_x\sin\theta + ru_y\cos\theta \\
\text{(iii)} & v_r = v_x x_r + u_y y_r = v_x\cos\theta + v_y\sin\theta \\
\text{(iv)} & v_\theta = v_x x_\theta + v_y y_\theta = -rv_x\sin\theta + rv_y\sin\theta
\end{array}$$

$$\begin{aligned}
\text{CR-eqtn} \underset{\text{in }(x,y)}{\Longrightarrow} \quad (1)\ ru_r &= ru_x\cos\theta + ru_y\sin\theta \\
&= rv_y\cos\theta - rv_x\sin\theta \\
&= v_\theta \\
(2)\ u_\theta &= -ru_x\sin\theta + ru_y\cos\theta \\
&= -r(u_x\sin\theta - u_y\cos\theta) \\
&= -r(v_y\sin\theta + v_x\cos\theta) \\
&= -rv_r
\end{aligned}$$

$\boxed{\text{CR-eqtn in polar form}}$: $\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$

Example: 1) $f(z) = r^{\frac{1}{n}} e^{\frac{i\theta}{n}}$, where $r > 0$, $\theta \in (-\pi, \pi)$

($z \mapsto$ principal n-th root of z)

$$f(z) = r^{\frac{1}{n}} e^{\frac{i\theta}{n}} = r^{\frac{1}{n}} \cos\frac{\theta}{n} + i r^{\frac{1}{n}} \sin\frac{\theta}{n}$$

$$\Rightarrow \begin{cases} u(r,\theta) = r^{\frac{1}{n}} \cos\frac{\theta}{n} \\ v(r,\theta) = r^{\frac{1}{n}} \sin\frac{\theta}{n} \end{cases}$$

$$\Rightarrow \begin{cases} u_r = \frac{1}{n} r^{\frac{1}{n}-1} \cos\frac{\theta}{n}, & v_r = \frac{1}{n} r^{\frac{1}{n}-1} \sin\frac{\theta}{n} \\ u_\theta = -\frac{r^{\frac{1}{n}}}{n} \sin\frac{\theta}{n}, & v_\theta = \frac{r^{\frac{1}{n}}}{n} \cos\frac{\theta}{n} \end{cases}$$

$$\Rightarrow \begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$$

$\Rightarrow$ f is complex differentiable.

Harmonic function

Defn: A function $f: \mathbb{R}^2 \to \mathbb{R}$ is harmonic if its Laplacian $\Delta f := f_{xx} + f_{yy}$ vanishes. i.e. $\Delta f \equiv 0$.

Let $f = u + iv$ be complex differentiable.

Then $\begin{cases} \Delta u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = 0 \\ \Delta v = v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = 0 \end{cases}$

Hence, $f = u + iv$ complex differentiable $\Rightarrow$ u & v harmonic.

Furthermore, u and v determine each other up to a constant.

Prop: If $\begin{cases} f_1 = u + iv_1 \\ f_2 = u + iv_2 \end{cases}$ are complex differentiable,

then $v_1 = v_2 + \text{constant}$.

<u>Pf</u>: f_i's complex differentiable

$\Rightarrow$ $f_1 - f_2$ complex differentiable.

$\Rightarrow$ CR-eqtn hold for $f_1 - f_2 = 0 + i(v_1 - v_2)$

$\Rightarrow$ $(v_1 - v_2)_x = 0 = (v_1 - v_2)_y$

$\Rightarrow$ $v_1 = v_2 +$ constant.